Repeated Root and Common Root

1. **(Method 1)**

Let α, β, γ be the roots of $p(x) = x^3 + ax + b = 0$ …. (1)

Then $\alpha + \beta + \gamma = 0$, $\alpha\beta + \beta\gamma + \gamma\alpha = a$, $\alpha\beta\gamma = -b$ …. (2)

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = (\alpha + \beta)^2 - 4[a - (\beta\gamma + \gamma\alpha)] = -4a + [(\alpha + \beta)^2 + \gamma(\alpha + \beta)] + 3\gamma(\alpha + \beta)$

$= -4a + [(\alpha + \beta + \gamma)(\alpha + \beta)] + 3\gamma(\alpha + \beta) = -4a + 0 + 3\gamma(-\gamma) = -4a - 3\gamma^2$

Similarly, $(\beta - \gamma)^2 = -4a - 3\alpha^2$ and $(\gamma - \alpha)^2 = -4a - 3\beta^2$

$\therefore (\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 = (-4a - 3\gamma^2)(-4a - 3\alpha^2)(-4a - 3\beta^2) = -(4a + 3\gamma^2)(4a + 3\alpha^2)(4a + 3\beta^2)$

$= -[64a^3 + 48a^2(\alpha^2 + \beta^2 + \gamma^2) + 36a(\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2) + 27\alpha^2\beta^2\gamma^2]$

$= -\{64a^3 + 48a^2[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)] + 36a[(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma) + 27\alpha^2\beta^2\gamma^2]$

$= -\{64a^3 + 48a^2[0 - 2a] + 36a[a^2 - 2(-b)(0)] + 27(-b)^2\}$

$= -\{4a^3 + 27b^2\}$

$\therefore$ (1) has 3 distinct roots $\Leftrightarrow -(\alpha - \beta)^2(\beta - \gamma)^2(\gamma - \alpha)^2 < 0 \Leftrightarrow 4a^3 + 27b^2 < 0$

(Method 2)

Let $p(x) = x^3 + ax + b$, $p'(x) = 3x^2 + a$, $p''(x) = 6x$

Since $\lim_{x \to +\infty} p(x) = +\infty$, $\lim_{x \to -\infty} p(x) = -\infty$, $p(x) = 0$ has at least one real root .

For stationary points, $p'(x) = 0$, $3x^2 + a = 0 \Rightarrow x = \pm\sqrt{-\frac{a}{3}}$ …. (1)

If $a > 0$, then there is no stationary point, but the point $(0, b)$ is a real point of inflexion.

$\therefore$ $p(x) = 0$ has only one real root and two complex roots.

If $a = 0$ and $b \neq 0$, there is a stationary point when $x = 0$ and $(0, b)$ is a real inflexion.

$\therefore$ $p(x) = 0$ has only one real root and two complex roots.

If $a = 0$ and $b = 0$, the origin is at the inflexion and $p(x) = 0$ has roots 0, 0, 0.

If $a < 0$, then by (1), $x = +\sqrt{-\frac{a}{3}}$ is a min. since $p''\left(+\sqrt{-\frac{a}{3}}\right) > 0$.

and $x = -\sqrt{-\frac{a}{3}}$ is a max. since $p''\left(-\sqrt{-\frac{a}{3}}\right) < 0$.

$\therefore$ $y_{min} = b + \frac{2a}{3}\sqrt{-\frac{a}{3}}$, $y_{max} = b - \frac{2a}{3}\sqrt{-\frac{a}{3}}$

$\therefore$ $p(x) = 0$ has 3 distinct roots $\Leftrightarrow y_{min} < 0$ and $y_{max} > 0$

$\Leftrightarrow b + \frac{2a}{3}\sqrt{-\frac{a}{3}} < 0$ and $b - \frac{2a}{3}\sqrt{-\frac{a}{3}} > 0 \Leftrightarrow \left(b + \frac{2a}{3}\sqrt{-\frac{a}{3}}\right)\left(b - \frac{2a}{3}\sqrt{-\frac{a}{3}}\right) < 0$

$\Leftrightarrow 27b^2 + 4a^3 < 0$

(Note : If $a < 0$, For two different real roots, one being repeated, the necessary and sufficient condition is $27b^2 + 4a^3 = 0$. For one real roots, two complex roots : $27b^2 + 4a^3 > 0$)

2. **(Method 1)**

Let $P(x) = 3x^5 + 2x^4 + x^3 - 6x^2 - 5x - 4$, then $P'(x) = 15x^4 + 8x^3 + 3x^2 - 12x - 5$

By the multiple root theorem, $P(\omega) = 3\omega^5 + 2\omega^4 + \omega^3 - 6\omega^2 - 5\omega - 4 = 0$, $P'(\omega) = 15\omega^4 + 8\omega^3 + 3\omega^2 - 12\omega - 5 = 0$

By Euclidean algorithm, (working steps not shown here),

H.C.F. $(P(x), P'(x)) = x^2 + x + 1$

$\therefore$ $P(x) = (x^2 + x + 1)^2 (3x - 4)$, by division (working steps not shown here) .

$\therefore$ The roots are $\dfrac{-1 \pm \sqrt{3}}{2}$ (double roots) and $\dfrac{4}{3}$.

(Method 2)

Since $P(x) = 0$ has a complex root ω, its conjugate $\overline{\omega}$ is also a root.

Thus $P(x) = 0$ has four roots, $\omega, \omega, \overline{\omega}, \overline{\omega}$. As a result, $P(x)$ has a factor : $(ax^2 + bx + c)^2$.

Since $\deg [P(x)] = 5$, there is a linear factor left.

Since the leading coefficient of $P(x)$ is 3, which is not a complete square, $\therefore$ $a = 1$.

Similarly, from the constant term $= -4$ of $P(x)$, $c = \pm 1$ or ± 2.

From the constant term $= -5$ of $P'(x)$, $c = \pm 2$ is rejected.

As a result, $P(x)$ has a factor : $(x^2 + bx \pm 1)^2$ and the linear factor is therefore $(3x \pm 4)$

On testing using the factor theorem $P(4/3) = 0$, and the linear factor is $(3x - 4)$

$\therefore$ $P(x) = (x^2 + bx + c)^2(3x - 4)$. On comparing coefficients, we can find $b = 1$, $c = 1$.

Result follows.

3. Put $p(z) = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + ... + \dfrac{z^n}{n!}$.

Since $p(0) \neq 0$. $\therefore$ $z = 0$ is not a root of $p(z)$.

$p(z) = 0 \Rightarrow 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + ... + \dfrac{z^{n-1}}{n!} = -\dfrac{z^n}{n!} \neq 0 \qquad \Rightarrow p'(z) = -\dfrac{z^n}{n!} \neq 0$

By the multiple root theorem, $p(z) = 0$ has no repeated roots .

4. Let $p(x) = (x - a_1)(x - a_3)(x - a_5) + k^2(x - a_2)(x - a_4)(x - a_6)$

$p(a_1) = k^2 (a_1 - a_2)(a_1 - a_4)(a_1 - a_6) < 0$, $\quad p(a_2) = (a_2 - a_1)(a_2 - a_3)(a_2 - a_5) > 0$

$p(a_4) = (a_4 - a_1)(a_4 - a_3)(a_4 - a_5) < 0$, $\quad p(a_6) = (a_6 - a_1)(a_6 - a_3)(a_6 - a_5) > 0$

Since $\deg [p(x)] = 3$ and there are 3 changes of sign as x increases from a_1 to a_6,

there are 3 distinct real roots .

5. Let $p(x) = x^4 - 4ax^3 + 6x^2 + 1$, $p'(x) = 4x^3 - 12a x^2 - 12x$

Since $p(0) = 1 \neq 0$. $\therefore$ $q = 0$ is not a root of $p(x) = 0$.

By the multiple root theorem, since q is the repeated root, $p'(q) = 4q^3 - 12a q^2 - 12q = 0$

$\therefore$ $4q (q^2 - 3aq + 3) = 0$.

Since $q \neq 0$, $\therefore$ $q^2 - 3aq + 3 = 0 \Rightarrow a = \dfrac{q^2 + 3}{3q}$ …. (1)

$p(q) = 0 \Rightarrow q^4 - 4a q^3 + 6q^2 + 1 = 0$ …. (2)

(1) ↓ (2), $q^4 - 4\left(\frac{q^2+3}{3q}\right)q^3 + 6q^2 + 1 = 0 \Rightarrow q^4 - 6q^2 - 3 = 0 \Rightarrow q^4 = 6q^2 + 3$

From (1), $a^4 = \left(\frac{q^2+3}{3q}\right)^4 = \frac{(q^4+6q^2+9)^2}{81q^4} = \frac{(6q^2+3+6q^2+9)^2}{81(6q^2+3)} = \frac{[12(q^2+1)]^2}{81(6q^2+3)} = \frac{144}{243}\frac{q^4+2q^2+1}{2q^2+1}$

$= \frac{16}{27}\frac{6q^2+3+2q^2+1}{2q^2+1} = \frac{16}{27}\frac{8q^2+4}{2q^2+1} = \frac{64}{27} = \left(\frac{4}{3}\right)^3$ $\therefore$ $a = \left(\frac{4}{3}\right)^{3/4}$.

6. **(Method 1)**

Let $p(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$

Since α is a multiple root of $p(x) = 0$, $1/\alpha$ is a multiple root of $p(1/y) = 0$

or $\left(\frac{1}{y}\right)^n + p_1\left(\frac{1}{y}\right)^{n-1} + \ldots + p_{n-1}\left(\frac{1}{y}\right) + p_n = 0$, $g(y) = p_ny^n + p_{n-1}y^{n-1} + \ldots + p_1 + 1 = 0$

By Multiple Root Theorem, $1/\alpha$ is a root of $g'(y) = 0$, i.e., $g'(y) = np_ny^{n-1} + (n-1)p_{n-1}y^{n-2} + \ldots + p_1 = 0$

$\therefore$ α is also a root of $np_n\left(\frac{1}{x}\right)^{n-1} + (n-1)p_{n-1}\left(\frac{1}{x}\right)^{n-2} + \ldots + p_1 = 0$, or

$p_1x^{n-1} + 2p_2x^{n-2} + 3p_3x^{n-3} + \ldots + (n-1)p_{n-1}x + np_n = 0.$

(Method 2)

Let $p(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$

$p'(x) = nx^{n-1} + (n-1)p_1x^{n-2} + (n-2)p_2x^{n-3} + \ldots + p_{n-1}$

α is a multiple root $\Rightarrow$ $p(\alpha) = p'(\alpha) = 0$ $\Rightarrow$ $np(\alpha) - \alpha p'(\alpha) = 0$

$\Rightarrow$ α is a root of $p_1x^{n-1} + 2p_2x^{n-2} + 3p_3x^{n-3} + \ldots + (n-1)p_{n-1}x + np_n = 0.$

7. **(a)** $a(h) = 0 \Rightarrow a(x) = (x-h)^mq(x)$, where $q(h) \neq 0$.

$\therefore$ $a'(x) = (x-h)^m q'(x) + m(x-h)^{m-1} q(x) = (x-h)^{m-1}[(x-h)q'(x) + m\,q(x)] = (x-h)^{m-1} g(x)$

where $g(x) = (x-h)q'(x) + m\,q(x)$ and $g(h) = m\,q(h) \neq 0$.

But $a'(x) = (x-h)^s f(x)$, where $g(h) \neq 0$, since h is an s-multiple root of $a'(x)$.

$\therefore$ $(x-h)^{m-1} g(x) = (x-h)^s f(x)$

If $s \neq m-1$, without lost of generality, we assume $s > m-1$

then $g(x) = (x-h)^{s-(m-1)} f(x)$ and therefore $g(h) = (h-h)^{s-(m-1)} f(h) = 0$, contradicting to $g(h) \neq 0$.

$\therefore$ $s = m-1$, $m = s+1$, and $a(x) = (x-h)^{s+1} q(x)$, where $q(h) \neq 0$.

$\therefore$ h is an $(s+1)$-multiple root of $a(x)$.

Converse:

h is an $(s+1)$-multiple root of $a(x)$ $\Rightarrow$ $a(x) = (x-h)^{s+1}q(x)$, $q(h) \neq 0$.

Obviously, $a(h) = 0$.

Also, $a'(x) = (x-h)^{s+1}q'(x) + (s+1)(x-h)^s q(x) = (x-h)^s[(x-h)q'(x) + (s+1)q(x)] = (x-h)^s g(x)$

$\therefore$ $g(h) = (h-h)q'(h) + (s+1)q(h) = (s+1)q(h) \neq 0$.

$\therefore$ h is an s-multiple root of $a'(x)$.

(b) Let $f(x) = ax^2 + bx + c$, $f'(x) = 2ax + b$.

$\therefore$ By (a), h is a double root of $f(x) = 0$ $\Leftrightarrow$ $f(h) = 0$ and $f'(h) = 0$

$\Leftrightarrow$ $ah^2 + ah + c = 0$ …. (1) and $2ah + b = 0$ …. (2)

From (2), $h = -\dfrac{b}{2a}$ …. (3)

(3)↓(1) , $a\left(-\dfrac{b}{2a}\right)^2 + b\left(-\dfrac{b}{2a}\right) + c = 0$ $\Leftrightarrow$ $\Delta = b^2 - 4ac = 0$

$\therefore$ $\Delta = b^2 - 4ac = 0$ $\Leftrightarrow$ h is a double root of $f(x) = 0$.

(c) If h is a triple root of $f(x) = ax^3 + 3bx^2 + 3cx + d$, then

$f(h) = ah^3 + 3bh^2 + 3ch + d = 0$ …. (4)

$f'(h) = 3ah^2 + 6bh + 3c = 0$ …. (5)

$f''(h) = 6ah + 6b = 0$ …. (6)

From (6), $h = -\dfrac{b}{a}$, subst. in (2), $a\left(-\dfrac{b}{a}\right)^2 + 2b\left(-\dfrac{b}{a}\right) + c = 0 \Rightarrow \dfrac{b}{a} = \dfrac{c}{b}$ …. (7)

(4) - $\dfrac{h}{3} \times (5)$, $bh^2 + 2ch + d = 0$

$\Rightarrow$ $b\left(-\dfrac{b}{a}\right)^2 + 2c\left(-\dfrac{b}{a}\right) + d = 0 \Rightarrow b\left(-\dfrac{c}{b}\right)^2 + 2c\left(-\dfrac{c}{b}\right) + d = 0 \Rightarrow \dfrac{c}{b} = \dfrac{d}{c}$ …. (8)

Result follows from (7) and (8).

(d) **(i)** $ax^3 + 3bx^2 + 3cx + d \equiv a(x - e)^2 (x - f)$, where $e \neq f$, $e \neq 0$.

$\equiv a(x^2 - 2ex + e^2)(x - f)$ $\equiv$ $ax^3 - a(2e + f)x^2 + a(e^2 + 2ef)x - ae^2f$

Comparing coeff. $3b = -a(2e + f)$ …. (9)

$3c = a(e^2 + 2ef)$ …. (10)

$d = -ae^2f$ …. (11)

(9) × (10) , $9bc = -a^2e(2e + f)(e + 2f)$

From (11), $9ad = -9a^2e^2f$

$ad = bc$ $\Rightarrow$ $9ad = 9bc$ $\Rightarrow$ $-a^2e(2e + f)(e + 2f) = -9a^2e^2f$ $\Rightarrow$ $(2e + f)(e + 2f) = 9ef$

$\Rightarrow$ $2e^2 + 5ef + 2f^2 = 9ef$ $\Rightarrow$ $2e^2 - 4ef + 2f^2 = 0$ $\Rightarrow$ $2(e - f)^2 = 0$ $\Rightarrow$ $e = f$

Contradiction, $\therefore$ $ad \neq bc$.

(ii) If k is a double root of $f(x) = 0$, then $1/k$ is a double root of $p(1/y) = 0$,

that is, $g(y) = dy^3 + 3cy^2 + 3by + a = 0$. $\therefore$ $g'(1/k) = 0$

$g'(y) = 3(dy^2 + 2cy + d)$ $\Rightarrow$ $3[d(1/k)^2 + 2c(1/k) + b] = 0$

$\Rightarrow$ $bk^2 + 2ck + d = 0$ …. (12)

(iii) If $f(x) = ax^3 + 3bx^2 + 3cx + d$ has a double root k , then:

$f(k) = ak^3 + 3bk^2 + 3ck + d = 0$ …. (13)

$f'(k) = 3ak^2 + 6bk + 3c = 0$ $\Rightarrow$ $ak^2 + 2bk + c = 0$ …. (14)

(9) × a, $abk^2 + 2ack + ad = 0$ …. (15)

(14)×b, $abk^2 + 2b^2k + bc = 0$ …. (16)

(15) – (16), $2(ac - b^2)k - (bc - ad) = 0 \Rightarrow k = \frac{1}{2} \times \frac{bc - ad}{ac - b^2}$ …. (17)

(iv) (14) × c, $ack^2 + 2bck + c^2 = 0$ …. (18)

(12) × b, $b^2k^2 + 2bck + bd = 0$ …. (19)

(18) – (19), $(ac - b^2)k^2 - (bd - c^2) = 0 \Rightarrow k^2 = \frac{bd - c^2}{ac - b^2}$ …. (20)

(17)↓(20), and move terms, $(bc - ad)^2 = 4(ac - b^2)(bd - c^2)$.

8. Let $p(x) = a_nx^n + \dots + a_1x + a_0 = 0$, If α is a repeated root of $p(x) = 0$, then

$p(x) = (x - \alpha)^2 g(x)$, where $g(x)$ is a polynomial.

$p'(x) = 2(x - \alpha) g(x) + (x - \alpha)^2 g'(x) = (x - \alpha) [2g(x) + (x - \alpha) g'(x)]$

$\therefore p'(\alpha) = 0 \Rightarrow \alpha$ is root of $p'(x) = na_nx^{n-1} + \dots + 2a_2 + a_1 = 0$.

Let $p(x) = 24x^4 - 20x^3 - 6x^2 + 9x - 2$, $p'(x) = 96x^3 - 60x^2 - 12x + 9 = 3(32x^3 - 20x^2 - 4x + 3)$

$p''(x) = 3(96x^2 - 40x - 4) = 12(24x^2 - 10x - 1)$

Now, $p''(x) = 0 \Rightarrow (2x - 1)(12x + 1) = 0 \Rightarrow x = 1/2$ or $-1/12$

Since $p(x) = 0$ has a triple root, this root must be a root of $p''(x) = 0$.

By trial, $p(1/2) = 0$, therefore 1/2 is the triple root.

By division, we get $p(x) = (2x - 1)^3 (3x + 2)$ and hence the roots are 1/2 (triple root) and -2/3.

9. First part is omitted.

(a) Let $f(x) = Ax^{n+1} + Bx^n + 1$. Since $f(x)$ is divisible by $(x - 1)^2$, $x = 1$ is a double root of $f(x) = 0$.

$\therefore f(1) = A + B + 1 = 0$ …. (1)

$f'(x) = (n + 1) Ax^n + n Bx^{n-1} \Rightarrow f'(1) = (n + 1)A + nB = 0$ …. (2)

From (1), $nA + nB = -n$ …. (3)

(3)↓(2), $-n + A = 0 \Rightarrow A = n$ …. (4)

(4)↓(3), $B = -1 - n$ $\therefore f(x) = nx^n+1 - (n + 1)x^n + 1$.

(b) See number 3.

10. Let $f(x) = x^4 + 12x^3 + 32x^2 - 24x + 4$

then $f'(x) = 4x^3 + 36x^2 + 64x - 24 = 4(x^3 + 9x^2 + 16x - 6) = 4(x^2 + 6x - 2)(x + 3)$

By trial $f(-3) \neq 0$. $\therefore x^2 + 6x - 2$ is a repeated factor of $f(x)$. [irrational roots occur in pairs]

$\therefore f(x) = (x^2 + 6x - 2)^2$

$\therefore f(x) = 0$ has roots $-3 \pm \sqrt{11}$ (repeated roots) .

11. Let $f(x) = x^6 - 5x^5 + 5x^4 + 9x^3 - 14x^2 - 4x + 8$

then $f'(x) = 6x^5 - 25x^4 + 20x^3 + 27x^2 - 28x - 4$, $f''(x) = 30x^4 - 100x^3 + 60x^2 + 54x - 28$

By trial $f(2) = f'(2) = f''(2) = 0$

$\therefore x = 2$ is the multiple roots of multiplicity 3 of $f(x) = 0$.

By division $f(x) = (x - 2)^3 (x^3 + x^2 - x - 1) = (x - 2)^3 [x^2(x + 1) - (x + 1)] = (x - 2)^3(x - 1)(x + 2)^2$.

The roots are 2 (triple root) , 1 and -1 (double roots) .

12. Let $f(x) = x^n - a^n \quad (a \neq 0)$, $f'(x) = nx^{n-1}$.

$f(x)$ has a double root at $x = r$ if $f(r) = 0$ and $f'(r) = 0$.

$\therefore \quad r^n - a^n = 0 \quad \ldots. \quad (1) \qquad nr^{n-1} = 0 \quad \ldots. \quad (2)$

From (2), $r = 0$ is the only possible multiple root.

Sub. $r = 0$ in (1), we get $a^n = 0 \Rightarrow a = 0$ contradicting to the given $a \neq 0$.

$\therefore$ $f(x) = 0$ cannot have repeated roots.

13. Let $f(x) = x^n + nx^{n-1} + n(n-1)x^{n-2} + \ldots + n!$, $f'(x) = nx^{n-1} + n(n-1)x^{n-2} + \ldots + n!$

$f(x)$ has a double root at $x = r$ if $f(r) = 0$ and $f'(r) = 0$.

$\therefore \quad f(r) = r^n + nr^{n-1} + n(n-1)r^{n-2} + \ldots + n! \quad \ldots. \quad (1), \qquad f'(r) = nr^{n-1} + n(n-1)r^{n-2} + \ldots + n! \quad \ldots. \quad (2)$

(1) – (2), $r^n = 0 \Rightarrow r = 0$.

But $f(0) = n! \neq 0$. $\therefore$ $r = 0$ is not a root of $f(x) = 0$. Contradiction.

$\therefore$ $f(x) = 0$ cannot have repeated roots.

14. Let $f(x) = x^4 + px^2 + q = 0 \quad \ldots. \quad (1)$

By multiple root theorem, $f'(x) = 4x^3 + 2px = 0 \quad \ldots. \quad (2)$

$f''(x) = 12x^2 + 2p = 0 \quad \ldots. \quad (3)$

$(3) \times x, \quad 12x^3 + 2px = 0 \quad \ldots. \quad (4)$

(4) – (2), $8x^3 = 0 \Rightarrow x = 0$ may be the only possible repeated root.

But , If $x = 0$, $f(0) = q = 0$ $\therefore f(x) = x^4 + px^2 = x^2(x^2 + p)$

If $p = 0$, $f(x) = 0$ has $x = 0$ of multiplicity 4.

If $p \neq 0$, $f(x) = 0$ has $x = 0$ of multiplicity 2 and $\pm\sqrt{-p}$ as roots.

In either case, the equation $x^4 + px^2 + q = 0$ cannot have exactly three equal roots.

15. $\begin{cases} ax^2 + bx + a = 0 \\ x^3 - 2x^2 + 2x - 1 = 0 \end{cases} \Rightarrow \begin{cases} x^2 + (b/a)x + 1 = 0 \quad \ldots.(1) \\ (x-1)(x^2 - x + 1) = 0 \ldots.(2) \end{cases}$

From (2), $x = 1$ is a root. The other two roots are complex and are conjugate roots.

(a) If (1) and (2) have exactly one root in common. Then the root in common must be 1.

Sub. $x = 1$ in (1) , $1 + (b/a)(1) + 1 = 0 \Rightarrow b/a = -2$.

and the equation (1) is $x^2 - 2x + 1 = 0 \Rightarrow (x-1)^2 = 0 \Rightarrow x = 1$.

(b) If (1) and (2) have exactly two roots in common, then $x^2 - x + 1 = 0$ from (2).

Compare coefficients with (1), $\therefore$ $b/a = -1$.

The common roots are $x = \dfrac{1 \pm \sqrt{3}i}{2}$.

16. $x^3 - x^2 + 6x + 24 = 0 \quad \ldots. \quad (1), \qquad x^2 - x + b = 0 \quad \ldots. \quad (2)$

$(1) - (2) \times x$, $(6 - b)x = -24 \Rightarrow x = 24/(6 - b) \quad \ldots. \quad (3)$

From (1), $x^3 - x^2 + 6x + 24 = (x + 2)(x^2 - 3x + 12) = 0$.

Since the roots of $x^2 - 3x + 12 = 0$ are complex and must occur in pairs, (1) and (2) have no complex common roots and the only common root is $x = -2$. Put $x = -2$ in (3), $b = -6$.

17. $2x^3 + 5x^2 - 6x - 9 = 0$ .... (1) , $3x^3 + 7x^2 - 11x - 15 = 0$ .... (2)

Let α, β, γ be the roots of (1) and α, β, δ be the roots of (2).

Then $\alpha + \beta + \gamma = -5/2$ .... (3) $\alpha\beta\gamma = 9/2$ .... (4)

$\alpha + \beta + \delta = -7/3$ .... (5) $\alpha\beta\delta = 15/3$ .... (6)

(3) – (5) , $\gamma - \delta = -1/6$ .... (7) (4)/(6), $\gamma = 9\delta/10$ .... (8)

(8)↓(7), $\therefore \delta = 5/3$ .... (9) (9) ↓(5), $\alpha + \beta = -4$ .... (10)

(9)↓(6), $\alpha\beta = 3$ .... (11) Solving (10), (11), $\alpha = -3$, $\beta = -1$.

18.

Let $f(x) = x^3 - 2x^2 - 2x + 1 = 0$ and $g(x) = x^4 - 7x^2 + 1 = 0$.

Apply Euclidean Algorithm, the H.C.F. of f(x) and g(x) is $x^2 - 3x + 1$.

$\therefore \quad x^2 - 3x + 1 = 0$

$\therefore \quad x = \dfrac{3 \pm \sqrt{5}}{2}$ are the common roots .

1	**1 +0 -7 +0 +1**	**1 -2 -2 +1**	**1**
	1 -2 -2 +1	**1 -3 +1**	
2	**2 -5 -1 +1**	**1 -3 +1**	**1**
	2 -4 -4 +2	**1 -3 +1**	
	-1 +3 -1		

19.

Let $f(x) = 6x^3 + 7x^2 - x - 2 = 0$ & $g(x) = 6x^4 + 19x^3 + 17x^2 - 2x - 6 = 0$.

Apply Euclidean Algorithm, the H.C.F. of f(x) and g(x) is $2x^2 + x - 1$.

$\therefore \quad 2x^2 + x - 1 = 0 \Rightarrow (2x + 1)(x + 1) = 0 \Rightarrow x = 1/2$ or -1 .

$\therefore \quad x = 1/2$ and -1 are the common roots .

3	**6 +7 -1 -2**	**6 +19 +17 -2 -6**	**1**
	6 +3 -3	**6 +7 -1 -2**	
2	**4 +2 -2**	**12 +18 +0 -6**	**2**
	4 +2 -2	**12 +14 -2 -4**	
		4 +2 -2	

20. **(a)** -1/2 (double root), 4.

(b) 3 (double root), 2 i , - 2i .

(c) 1/2 (triple root), -3/2 .

21. **(a)** $\begin{cases} f(x) = x^2 + ax + b = 0 \\ g(x) = x^2 + a'x + b' = 0 \end{cases}$ has a common root .

$\Leftrightarrow \begin{cases} f(x) - g(x) = (a - a')x + (b - b') = 0 \\ ag(x) - a'f(x) = (a - a')x^2 + (ab' - a'b) = 0 \end{cases}$ has a common root .

$\Leftrightarrow x = -\dfrac{b - b'}{a - a'}$ and $x^2 = -\dfrac{ab' - a'b}{a - a'}$

$\Leftrightarrow \left(-\dfrac{b - b'}{a - a'}\right)^2 = -\dfrac{ab' - a'b}{a - a'}$

$\Leftrightarrow \quad (b - b')^2 + (a - a')(ab' - a'b) = 0$ .... (1)

(b) $\begin{cases} 2x^2 - (k + 2)x + 12 = 0 \\ 4x^2 - (3k - 2)x + 36 = 0 \end{cases} \Rightarrow \begin{cases} x^2 - \left(\dfrac{k + 2}{2}\right)x + 6 = 0 \\ x^2 - \left(\dfrac{3k - 2}{4}\right) + 9 = 0 \end{cases} \Rightarrow \begin{cases} a = -\left(\dfrac{k + 2}{2}\right), \; b = 6 \\ a' = -\left(\dfrac{3k - 2}{4}\right), \; b' = 9 \end{cases}$

Substitute in (1) and solve for k, $\therefore \quad k = 9$.